The purpose of this note is to further generalize the *smoothness framework* by Roughgarden [7], in order to provide a unified proof for many existing *price-of-anarchy* results.

1 Bounds on the Price of Anarchy

A strategic *game* is a triple $(N, (S_i)_{i \in N}, (h_i)_{i \in N})$, where N is the set of players, S_i is the set of (pure) strategies, $h_i : S \to \mathbb{R}$ is the payoff function of player i, and S denotes the Cartesian product $S = \times_{i \in N} S_i$.

We assume that the quality of strategy profiles—the outcomes of a game—can be measured by a single non-negative quantity. We call a map $\Psi : S \to \mathbb{R}_{\geq 0}$ *social welfare* function. The *price of anarchy* is the worst-case ratio between the social welfare of an equilibrium and that of a maximum-payoff strategy profile. Of course, all definitions can be easily restated for *cost-minimization* games where each player seeks to minimize its cost c_i instead of maximizing its payoff h_i. In this case, Ψ is called the *social cost* function. Obviously, the price of anarchy is always at most 1 for payoff-maximization games and at least 1 for cost-minimization games.

Strategy profiles are sometimes a verbose way of capturing a game's state and include levels of detail unneeded or even cumbersome for the analysis of game properties. Therefore, our framework allows for partitioning the set of strategy profiles S into more convenient equivalence classes. We will call such an equivalence class *allocation* and require that the set of allocations A is granular enough to separate equilibrium profiles from non-equilibrium profiles (we do not make any other a priori assertions). When unambiguous, we use equivalence classes in the same way as strategy profiles.

1.1 The Smoothness Framework

The core of the smoothness framework consists of decoupling the sum of the social welfare function plus a variational inequality, where the latter completely comprises all equilibrium properties needed for an upper bound.

Definition 1.1. *Let $\mathcal{P}$ be a set of probability measures on the set of allocations A. We say that a function $\Delta : A \times A \to \mathbb{R}$ is a* separator *for $\mathcal{P}$ if there is some optimal (pure) allocation y such that for every $P \in \mathcal{P}$ it holds that $\mathrm{E}_{X \sim P}[\Delta(X, y)] \geq 0$.*

Definition 1.2. *Let $\lambda \geq 0$, $\mu < 1$, and $\Delta : A \times A \to \mathbb{R}$. A payoff-maximization game is (λ, μ, Δ)-*smooth *if there is some optimal allocation $y \in A$ such that for every allocation $x \in A$,*

$$\Psi(x) - \Delta(x, y) \geq \mu \cdot \Psi(x) + \lambda \cdot \Psi(y) .$$

*Likewise, a cost-minimization game is (λ, μ, Δ)-*smooth *if there is some $y \in A$ such that for all $x \in A$,*

$$\Psi(x) + \Delta(x, y) \leq \mu \cdot \Psi(x) + \lambda \cdot \Psi(y) .$$

Clearly, if a payoff-maximization game is (λ, μ, Δ)-smooth and Δ is a separator for a certain set of equilibria (such as pure Nash, mixed Nash, correlated, etc.), then every equilibrium P in that set has expected social welfare at least $\lambda/(1-\mu)$ times that of a (pure) optimal allocation y. In detail, using the linearity of expectation,

$$\begin{aligned} \mathrm{E}_{X\sim P}\Big[\Psi(X)\Big] &\geq \mathrm{E}_{X\sim P}\Big[\Psi(X)-\Delta(X,y)\Big] \geq \mathrm{E}_{X\sim P}\Big[\mu\cdot\Psi(X)+\lambda\cdot\Psi(y)\Big] \\ &= \mu\cdot \mathrm{E}_{X\sim P}\Big[\Psi(X)\Big]+\lambda\cdot\Psi(y)\,, \end{aligned} \tag{1.1}$$

where rearranging terms yields the claimed bound. It is immediate how this calculation can be mimicked for cost-minimization games.

1.2 Δ-Separation for Specific Classes of Games

Looking at (1.1) shows that the generality of bounds derived via the smoothness framework hinges on the separation provided by Δ; i.e., a bound carries over to the most general solution concept for which Δ is a separator. In this section, we define separators Δ for several natural classes of games.

1.2.1 Finite Number of Players

If the number of players is finite, a straightforward and general way to define a separator Δ is to make use of the Nash conditions.

Lemma 1.3. *Suppose N is finite. Let $A = S$ and define $\Delta(\vec{x},\vec{y}) := \sum_{i\in N}\big(h_i(\vec{x}) - h_i(\vec{x}^{-i},\vec{y}^i)\big)$. Then, Δ is a separator for coarse correlated equilibria.*

Proof. For every coarse correlated equilibrium P and any (pure) allocation $\vec{x}$ we have

$$\mathrm{E}_{\vec{X}\sim P}[\Delta(\vec{X},\vec{y})] = \sum_{i\in N}\big(\mathrm{E}_{\vec{X}\sim P}[h_i(\vec{x})] - \mathrm{E}_{\vec{X}\sim P}[h_i(\vec{x}^{-i},\vec{y}^i)]\big) \geq 0\,. \qquad \square$$

1.2.2 Finite Number of Players, Convex Strategy Sets

If the number of players is finite, the strategy sets S_i are convex subsets of some Euclidean space $\mathbb{R}^{m_i}$, and player payoff functions are continuously differentiable with a bounded derivative (i.e., in $C^1(S)$), we can define a separator Δ using the first-order necessary conditions for Nash equilibria. In detail, if $\vec{x}\in S$ is a Nash equilibrium, then it must satisfy the following variational inequalities,

$$\forall i\in N : \forall \vec{y}^i \in S_i : \langle \nabla_i h_i(\vec{x}), \vec{x}^i - \vec{y}^i\rangle \geq 0\,. \tag{1.2}$$

Here, $\nabla_i h_i := (\partial h_i/\partial x^i_1,\ldots,\partial h_i/\partial x^i_{m_i})$ denotes the gradient of h_i with respect to $\vec{x}^i$. Otherwise, due to continuity of the h_i, there would be some $\epsilon > 0$, a player i, and a strategy $\vec{y}^i$ so that player i could improve by playing $(1-\epsilon)\cdot\vec{x}^i + \epsilon\cdot\vec{y}^i$ instead.

We say a game is *concave* if all $\vec{x}^i \mapsto h_i(\vec{x}^{-i},\vec{x}^i)$ are concave functions. For concave games condition (1.2) is not only necessary but also sufficient, since it holds for all i, $\vec{x}$, and $\vec{y}$ that $h_i(\vec{x}) - h_i(\vec{x}^{-i},\vec{y}^i) \geq \langle\nabla_i h_i(\vec{x}), \vec{x}^i - \vec{y}^i\rangle$. Hence, a separator based on the first-order necessary condition for Nash equilibria can give a "sharper" separator than Lemma 1.3, potentially leading to a better price-of-anarchy bound.

We first show that the characterization by variational inequalities can be generalized to correlated equilibria, using ideas by Neyman [4]. Afterwards, we give the corresponding separator based on (1.2). It turns out that this separator does not work for coarse correlated equilibria.

Lemma 1.4. *Suppose N is finite. For every player i, every strategy profile $\vec{x} \in S$, and every strategy $\vec{y}^i \in S_i$, define*

$$\Delta_i(\vec{x}, \vec{y}^i) := \langle \nabla_i h_i(\vec{x}), \vec{x}^i - \vec{y}^i \rangle \,.$$

Let P be a probability measure on the set of pure strategy profiles. If P is a correlated equilibrium then it holds for every player i and every measurable function $f : S_i \to S_i$ that $\mathrm{E}_{\vec{X} \sim P}[\Delta_i(\vec{x}, f(\vec{X}^i))] \geq 0$. If the game is concave, this condition is also sufficient.

Proof. We start by proving necessity ("$\Rightarrow$"). Let P be a correlated equilibrium, and fix a player i and a measurable function $f : S_i \to S_i$. By way of contradiction, suppose $\mathrm{E}_{\vec{X} \sim P}[\Delta_i(\vec{X}, f(\vec{X}^i))] < 0$. For any $\vec{x} \in S$ and any $\epsilon \in [0,1]$, we define the abbreviating notation $\vec{x} \diamond \epsilon := (\vec{x}^{-i}, (1-\epsilon) \cdot \vec{x}^i + \epsilon \cdot f(\vec{x}^i))$. Note that

$$\lim_{\epsilon \searrow 0} \frac{h_i(\vec{x}) - h_i(\vec{x} \diamond \epsilon)}{\epsilon} = \Delta_i(\vec{x}, f(\vec{x}^i)) \,.$$

Moreover, the fraction in the limit is bounded (independent of $\vec{x} \in S$) by a constant function, which is trivially P-integrable. Thus, by the dominated convergence theorem,

$$\lim_{\epsilon \searrow 0} \int \frac{h_i(\vec{x}) - h_i(\vec{x} \diamond \epsilon)}{\epsilon} \, dP(\vec{x}) = \int \Delta_i(\vec{x}, f(\vec{x}^i)) \, dP(\vec{x}) = \mathrm{E}_{\vec{X} \sim P}[\Delta_i(\vec{X}, f(\vec{X}^i))] < 0 \,.$$

Hence there exists $\epsilon > 0$ so that

$$\mathrm{E}_{\vec{X} \sim P}[h_i(\vec{X})] = \int h_i(\vec{x}) \, dP(\vec{x}) < \int h_i(\vec{x} \diamond \epsilon) \, dP(\vec{x}) = \mathrm{E}_{\vec{X} \sim P}[h_i(\vec{X} \diamond \epsilon)] \,,$$

in contradiction to the assumption that P is a correlated equilibrium.

We now show sufficiency ("$\Leftarrow$") in concave games. In particular, such a game satisfies for every strategy profile $\vec{x} \in S$, every player i, and every measurable function $f : S_i \to S_i$ that $h_i(\vec{x}) - h_i(\vec{x}^{-i}, f(\vec{x}^i)) \geq \Delta_i(\vec{x}, f(\vec{x}^i))$. Hence,

$$\mathrm{E}_{\vec{X} \sim P}[h_i(\vec{X}) - h_i(\vec{X}^{-i}, f(\vec{X}^i))] \geq \mathrm{E}_{\vec{X} \sim P}[\Delta_i(\vec{X}, f(\vec{X}^i))] \geq 0 \,,$$

which shows that P is a correlated equilibrium. □

Corollary 1.5. *Suppose N is finite. Let $A = S$ and define $\Delta(\vec{x}, \vec{y}) := \sum_{i \in N} \langle \nabla_i h_i(\vec{x}), \vec{x}^i - \vec{y}^i \rangle$. Then, Δ is a separator for correlated equilibria.*

Remark 1.6. Neither Lemma 1.4 nor Corollary 1.5 generalize to coarse correlated equilibria. Moreover, the price of anarchy in games with convex strategy sets is generally strictly worse for coarse correlated equilibria than for correlated equilibria.

We give an example in the following. Consider the cost-minimization game defined by $N = \{1, 2\}$, $S_1 = S_2 = [0,1]$, and $c_1(\vec{x}) = c_2(\vec{x}) = d(\vec{x}) + \varepsilon$, where $d(\vec{x}) := (x_1 - x_2)^2$ and $\varepsilon > 0$. Note that the game is an identical-interest game and has concave payoff/convex cost functions. The sole purpose of ε is to ensure that the optimum has strictly positive social cost. Let P be a randomized strategy profile that chooses $(0, \alpha)$ and $(1, 1-\alpha)$ with equal probability, where $\alpha > 0$.

Note that

$$\begin{aligned} \mathrm{E}_{\vec{X}\sim P}[d(\vec{X})] &= \alpha^2\,, \\ \mathrm{E}_{\vec{X}\sim P}[d(X_1, y_2)] &= (y_2 - \tfrac{1}{2})^2 + \tfrac{1}{4}\,, \\ \mathrm{E}_{\vec{X}\sim P}[d(y_1, X_2)] &= (\alpha - \tfrac{1}{2})^2 + (y_1 - \tfrac{1}{2})^2\,. \end{aligned}$$

Hence, P is a coarse correlated equilibrium if $\alpha \leq \frac{1}{4}$; its expected social cost is $\mathrm{E}_{\vec{X}\sim P}[\Psi(\vec{X})] = 2\alpha^2 + 2\varepsilon$. Now, fix $\vec{y} = (\frac{1}{2}, \frac{1}{2})$. Clearly, $\vec{y}$ is an optimal strategy profile. Note that the gradients of h are

$$\nabla_1 h_1(\vec{x}) = \frac{\partial h_1(\vec{x})}{\partial x_1} = 2x_2 - 2x_1 \quad \text{and} \quad \nabla_1 h_1(\vec{x}) = \frac{\partial h_2(\vec{x})}{\partial x_2} = 2x_1 - 2x_2\,,$$

so we have $\mathrm{E}_{\vec{X}\sim P}[\nabla_1 h_1(\vec{X}) \cdot (X_1 - y_1)] = -2\alpha$ and $\mathrm{E}_{\vec{X}\sim P}[\nabla_2 h_2(\vec{X}) \cdot (X_2 - y_2)] = 2\alpha - 4\alpha^2$. This shows that neither Lemma 1.4 nor Corollary 1.5 generalizes to coarse correlated equilibria. Moreover,

$$\begin{aligned} \Delta(\vec{x}, \vec{y}) &= -2(x_1 - x_2) \cdot [(x_1 - y_1) - (x_2 - y_2)] \\ &= -2(x_1 - x_2)^2 + 2(x_1 - x_2)(y_1 - y_2) = -\Psi(\vec{x}) + 2\varepsilon = -\Psi(\vec{x}) + \Psi(\vec{y})\,, \end{aligned}$$

where the last two equalities hold due to optimality of $\vec{y}$. Consequently, the game is $(1, 0, \Delta)$-smooth, i.e., has a correlated price of anarchy of 1.

1.2.3 Anonymous Nonatomic Games with Finite Strategies

Suppose N is some interval of the reals, and $\overline{S} := \bigcup_{i \in N} S_i$ is finite. Any strategy profile $\vec{x}$ is a Lebesgue-measurable function $i \mapsto \vec{x}^i$. All players have the same payoff function that depends only on the player's own strategy and the total mass of the players playing each of the strategies.

In this setting, it will be convenient to define the set of allocations as the set of "mass distribution vectors". That is, two strategy profiles $\vec{x}, \vec{y}$ are equivalent if

$$(\lambda(\{i : \vec{x}^i = s\}))_{s \in \overline{S}} = (\lambda(\{i : \vec{y}^i = s\}))_{s \in \overline{S}}\,,$$

where λ is the Lebesgue measure. The set of allocations is thus $\mathbb{R}^{\overline{S}}_{\geq 0}$. Moreover, there is a continuous function $h : \overline{S} \times A \to \mathbb{R}$, so that for each player i it holds that $h_i(\vec{x}) = h(\vec{x}^i, [\vec{x}])$, where $[\vec{x}]$ denotes the equivalence class (i.e., allocation) $\vec{x}$ is in. We say an allocation is a *Wardrop equilibrium* if the payoff on each used strategy is greater or equal than the payoff on any other strategy.

Corollary 1.7. *Suppose N is some interval of the reals. Let $A = \mathbb{R}^{\overline{S}}_{\geq 0}$ and define $\Delta(\vec{x}, \vec{y}) := \sum_{s \in \overline{S}} h(s, \vec{x}) \cdot (x_s - y_s)$. Then, Δ separates Wardrop equilibria.*

2 Examples

We now apply the smoothness framework to four concrete classes of games: atomic splittable congestion games, atomic unsplittable congestion games, nonatomic congestion games, and resource allocation games.

2.1 Atomic Splittable Congestion Games

Atomic splittable congestion games are concave games, for which we will apply the Δ-separation from Section 1.2.2. The smoothness framework boils down to identifying a suitable pair of parameters λ, μ that satisfy

$$y \cdot \ell(x) + \kappa(x,y) \cdot \ell'(x) \leq \lambda \cdot y \cdot \ell(y) + \mu \cdot x \cdot \ell(x) \tag{2.1}$$

for every $\ell \in \mathcal{L}$ and $x, y \geq 0$, where $\kappa(x,y)$ is defined as $y^2/4$ if $x \geq y/2$ and $x(y-x)$ otherwise.

Proposition 2.1. *Let $n \in \mathbb{N}$ and $x, y \geq 0$. For all $\vec{x}, \vec{y} \in \mathbb{R}^n_{\geq 0}$ with $\sum_i x_i = x$ and $\sum_i y_i = y$ it holds that $\sum_i \left(y_i \cdot x_i - x_i^2\right) \leq \kappa(x,y)$.*

Theorem 2.2. *Let $\mathcal{L}$ be a set of cost functions. If $\mu < 1$, $\lambda \in \mathbb{R}$ satisfy (2.1) for all $\ell \in \mathcal{L}$ and $x, y \geq 0$, and Δ is the separator from Corollary 1.5, then any atomic splittable congestion game with cost functions in $\mathcal{L}$ is (λ, μ, Δ)-smooth.*

Proof. Let $\vec{x}$, $\vec{y}$ be pure strategy profiles. We have

$$\Delta(\vec{x}, \vec{y}) = \sum_{i \in N} \sum_{e \in E} \ell_e^i(\vec{x}_e) \cdot (y_e^i - x_e^i)\,.$$

Suppose now that $\vec{x}$ is a Nash equilibrium. Then,

$$\begin{aligned}
\Psi(\vec{x}) + \Delta(\vec{x}, \vec{y}) &= \sum_{i \in [n]} \sum_{e \in E} \left(x_e^i \cdot \ell_e(x_e) + y_e^i \cdot \ell_e^i(\vec{x}_e) - x_e^i \cdot \ell_e^i(\vec{x}_e)\right) \\
&= \sum_{e \in E} \left(y_e \cdot \ell(x_e) + \ell'_e(x_e) \cdot \sum_{i \in [n]} \left(y_e^i \cdot x_e^i - (x_e^i)^2\right)\right) \\
&\leq \sum_{e \in E} \left(y_e \cdot \ell(x_e) + \kappa(x_e, y_e) \cdot \ell'(x_e)\right) && (2.2) \\
&\leq \sum_{e \in E} \left(\mu \cdot x_e \cdot \ell_e(x_e) + \lambda \cdot y_e \cdot \ell_e(y_e)\right) && (2.3) \\
&= \mu \cdot \Psi(\vec{x}) + \lambda \cdot \Psi(\vec{y})\,,
\end{aligned}$$

where inequality (2.2) is due to Proposition 2.1, and inequality (2.3) is due to (2.1). ∎

Corollary 2.3. *Suppose the assertions of Theorem 2.2 hold. Then, the correlated price of anarchy of atomic splittable congestion games with cost functions from $\mathcal{L}$ is at most $\frac{\lambda}{1-\mu}$.*

Remark 2.4. While the upper bound for atomic *unsplittable* congestion games (from Section 2.2) also holds for atomic splittable games, we conjecture that the exact value is strictly less.

2.2 Atomic Unsplittable Congestion Games

For atomic unsplittable congestion games, we can apply the Δ-separation from Section 1.2.1. The smoothness framework boils down to identifying a suitable pair of parameters λ, μ that satisfy

$$y \cdot \ell(x+1) \leq \mu \cdot x \cdot \ell(x) + \lambda \cdot y \cdot \ell(y) \tag{2.4}$$

for every $\ell \in \mathcal{L}$ and $x, y \geq 0$ (see also [2, Theorem 1], [1, Theorem 1], and [7, Proposition 3.2]).

Proposition 2.5. *If $\mu < 1$, $\lambda \in \mathbb{R}$ satisfies* (2.4) *for all $\ell \in \mathcal{L}$ and $x, y \geq 0$, and Δ is the separator from Lemma 1.3, then any atomic unsplittable congestion game with cost functions in $\mathcal{L}$ is (λ, μ, Δ)-smooth.*

Proof. Let $\vec{x}$ be a Nash equilibrium, and $\vec{y}$ be a strategy profile. We have

$$\begin{aligned}\Psi(\vec{x}) + \Delta(\vec{x}, \vec{y}) &= \sum_{i \in N} c_i(\vec{x}^{-i}, y^i) \leq \sum_{e \in E} y_e \cdot \ell_e(x_e + 1) \leq \sum_{e \in E} \big[\mu \cdot x \cdot \ell_e(x) + y \cdot \ell_e(y)\big] \\ &= \mu \cdot \Psi(\vec{x}) + \lambda \cdot \Psi(\vec{y}) \,.\end{aligned}$$

□

Corollary 2.6. *Suppose the assertions of Proposition 2.5 hold. Then, the coarse-correlated price of anarchy of atomic unsplittable congestion games with cost functions from $\mathcal{L}$ is at most $\frac{\lambda}{1-\mu}$.*

We remark that the smoothness framework can be applied in the same way to atomic unsplittable congestion games with weighted players. Since this adaption is straightforward, it is omitted here.

2.3 Non-atomic Congestion Games

For non-atomic congestion games, we can apply the Δ-separation from Section 1.2.3. The smoothness framework boils down to identifying a suitable parameter μ that satisfies

$$y \cdot \ell(x) \leq \mu \cdot x \cdot \ell(x) + y \cdot \ell(y) \tag{2.5}$$

for every $\ell \in \mathcal{L}$ and $x, y \geq 0$ (see also [5, Section 3.3] and [3, Theorem 3.6]).

Proposition 2.7. *If $\mu < 1$ satisfies* (2.5) *for all $\ell \in \mathcal{L}$ and $x, y \geq 0$, and Δ is the separator from Corollary 1.7, then any non-atomic congestion game with cost functions in $\mathcal{L}$ is $(1, \mu, \Delta)$-smooth.*

Proof. Let $\vec{x}, \vec{y} \in A$ be allocations. We have $\Delta(\vec{x}, \vec{y}) = \sum_{e \in E} \ell_e(x_e) \cdot (y_e - x_e)$. Suppose now that $\vec{x}$ is a Wardrop equilibrium. Then,

$$\Psi(\vec{x}) + \Delta(\vec{x}, \vec{y}) = \sum_{e \in E} y_e \cdot \ell_e(x_e) \leq \sum_{e \in E} \big[\mu \cdot x_e \cdot \ell_e(x_e) + y_e \cdot \ell_e(y_e)\big] = \mu \cdot \Psi(\vec{x}) + \Psi(\vec{y}) \,.$$

□

Corollary 2.8. *Suppose the assertions of Proposition 2.7 hold. Then, the price of anarchy of nonatomic congestion games with cost functions from $\mathcal{L}$ is at most $\frac{1}{1-\mu}$.*

2.4 Resource-Allocation Games

There is a finite number of players, and the strategy of each player $i \in N$ is a non-negative bid b_i, i.e., $S_i = \mathbb{R}_{\geq 0}$. At strategy profile $\vec{b}$, each player i gets the amount

$$x_i = \frac{b_i}{\sum_{j \in N} b_j} \cdot C \tag{2.6}$$

of the shared resource. If $\vec{b} = \vec{0}$, then every player i gets $x_i = \frac{C}{|N|}$. (Note the difference to Roughgarden [6] in order to avoid case analyses in the following proofs.) The payoff of player i is $h_i(\vec{b}) = U_i(x_i) - b_i$, where U_i is concave, strictly increasing, and continuously differentiable. It is obvious that $\vec{0}$ cannot be a Nash equilibrium.

The set of allocations is $A := \{\vec{x} \in \mathbb{R}^N_{\geq 0} \mid \sum_{i \in N} x_i = C\}$, i.e., two strategy profiles are equivalent if they lead to the same resource allocation.

Lemma 2.9. *Given a resource-allocation game, define*

$$\Delta(\vec{x}, \vec{y}) := \sum_{i \in N} \left[U_i'(x_i) \cdot \left(1 - \frac{x_i}{C}\right) \cdot (x_i - y_i) \right].$$

Then, Δ separates pure Nash equilibria.

Proof. Let $\vec{b} \in S$ be a pure Nash equilibrium, define $\vec{x}(\vec{b}) \in A$ as the allocation corresponding to bid vector $\vec{b}$ and let $\vec{y} \in A$ be an arbitrary allocation. Define $\vec{d} \in S$ by

$$d_i := y_i \cdot \frac{\sum_j b_j}{C}.$$

The gradients of h are

$$\nabla_i h_i(\vec{b}) = U_i'\left(x_i(\vec{b})\right) \cdot \frac{\partial x_i(\vec{b})}{\partial b_i} - 1 = U_i'\left(x_i(\vec{b})\right) \cdot \frac{C - x_i(\vec{b})}{\sum_j b_j} - 1.$$

Consequently,

$$\begin{aligned} 0 \leq \sum_{i \in N} \nabla_i h_i(\vec{b}) \cdot (b_i - d_i) &= \sum_{i \in N} U_i'\left(x_i(\vec{b})\right) \cdot \frac{C - x_i(\vec{b})}{\sum_j b_j} \cdot (b_i - d_i) \\ &= \sum_{i \in N} U_i'\left(x_i(\vec{b})\right) \cdot \left(1 - \frac{x_i(\vec{b})}{C}\right) \cdot \left(x_i(\vec{b}) - y_i\right). \end{aligned}$$

Here, the inequality is due to the first-order necessary conditions for Nash equilibria (1.2), the middle equality is due to $\sum_j d_j = \sum_j b_j$, and the last equality is due to $x_i(\vec{b}) = \frac{b_i}{\sum_j b_i} \cdot C$ and $y_i = \frac{d_i}{\sum_j b_i} \cdot C$. ◻

The smoothness framework boils down to identifying a suitable $\lambda \in \mathbb{R}$ that satisfies

$$U(x) - U'(x) \cdot \left(1 - \frac{x}{C}\right)(x - y) \geq \lambda \cdot U(y) \tag{2.7}$$

for every non-negative concave function U and $x, y \geq 0$. We can choose $\lambda = \frac{3}{4}$ to obtain:

Proposition 2.10. *If Δ is the separator from Lemma 2.9, then any resource allocation game is $(\frac{3}{4}, 0, \Delta)$-smooth.*

Proof. Let $\vec{x}$ be a Nash allocation, and $\vec{y}$ be any other allocation. We have

$$\Psi(\vec{x}) - \Delta(\vec{x}, \vec{y}) = \sum_{i \in N} \left[U_i(x_i) - U_i'(x_i) \cdot \left(1 - \frac{x_i}{C}\right)(x_i - y_i)\right] \geq \sum_{i \in N} \lambda \cdot U_i(y_i) = \lambda \cdot \Psi(\vec{y}),$$

where the inequality is due to Roughgarden [6, Lemma 3.14]. ◻

Corollary 2.11. *The price of anarchy of resource-allocation games is at least $\frac{3}{4}$.*

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